

# A note on the function for the robust control charts (`rcc`) in the `rQCC` package

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## Abstract

In this note, we provide a brief summary of variables control charts and a description of how they are constructed using the `rcc` function in the robust quality control chart (`rQCC`) R package. Using `rcc` function, one can construct the traditional Shewhart-type variables control charts. In addition, using various robust location and scale estimates provided by the `rQCC` package, one can easily obtain robust alternatives to the traditional charts.

## 1 Introduction

Control charts, also known as Shewhart control charts [1, 2, 3], have been widely used to monitor whether a manufacturing process is in a proper state of control or not. The traditional Shewhart-type control charts are made up of the upper control limit (UCL), the center line (CL) and the lower control limit (LCL) and they have the form of  $CL \pm g \cdot SE$ , where the American Standard is based on  $g = 3$  with a target false alarm rate of 0.027% and the British Standard is based on  $g = 3.09$  with a target false alarm rate of 0.020%. The UCL is given by  $CL + g \cdot SE$  and the LCL is  $CL - g \cdot SE$ .

In what follows, we provide how to construct the traditional Shewhart-type control charts and robust alternatives to them using various robust location and scale estimates provided by the `rQCC` package. In this note, we assume that we have  $m$  samples and that each sample has the same sample size of  $n$ . Let  $X_{ij}$  be the  $i$ th sample (subgroup) from a stable manufacturing process, where  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ . We also assume that  $X_{ij}$  are independent and identically distributed as normal with mean  $\mu$  and variance  $\sigma^2$ . The  $A$ ,  $B$  and  $D$  notations here follow the definitions in ASTM (STP 15-C) [4] and ASTM (STP 15-D) [5].

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## 2 The $\bar{X}$ chart

In order to construct the  $CL \pm g \cdot SE$  control limits, we consider the relation

$$\frac{\bar{X}_k - E(\bar{X}_k)}{SE(\bar{X}_k)} = \pm g.$$

Since  $E(\bar{X}_k) = \mu$  and  $SE(\bar{X}_k) = \sigma/\sqrt{n_k}$ , we have

$$E(\bar{X}_k) \pm g \cdot SE(\bar{X}_k) = \mu \pm \frac{g}{\sqrt{n_k}}\sigma.$$

Then the control limits for the  $\bar{X}$  chart with the sample size  $n_k$  are given by

$$\begin{aligned} \text{UCL} &= \mu + A(n_k)\sigma, \\ \text{CL} &= \mu, \\ \text{LCL} &= \mu - A(n_k)\sigma, \end{aligned}$$

where  $A(n_k) = g/\sqrt{n_k}$ . In practice, the values of the parameters,  $\mu$  and  $\sigma$ , are not known. Thus, with the estimates  $\hat{\mu}$  and  $\hat{\sigma}$ , we have

$$\begin{aligned} \text{UCL} &= \hat{\mu} + \frac{g}{\sqrt{n_k}}\hat{\sigma}, \\ \text{CL} &= \hat{\mu}, \\ \text{LCL} &= \hat{\mu} - \frac{g}{\sqrt{n_k}}\hat{\sigma}. \end{aligned} \tag{1}$$

Thus, we need to estimate  $\mu$  and  $\sigma$  by using each sample and then pooling these estimates. Using the  $i$ th sample above, the sample mean and variance are given by

$$\bar{X}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij} \quad \text{and} \quad S_i^2 = \frac{1}{n_i - 1} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2,$$

where  $i = 1, 2, \dots, m$ . Then we can estimate  $\mu$  using all the samples as below:

$$\bar{\bar{X}} = \frac{\bar{X}_1 + \bar{X}_2 + \dots + \bar{X}_m}{m} = \frac{1}{m} \sum_{i=1}^m \bar{X}_i.$$

Note that it is easily seen that  $\bar{\bar{X}}$  is unbiased for  $\mu$ . However,  $S_i$  is not unbiased for  $\sigma$  since  $E(S_i) = c_4(n_i)\sigma$ , where

$$c_4(n_i) = \sqrt{\frac{2}{n_i - 1}} \cdot \frac{\Gamma(n_i/2)}{\Gamma(n_i/2 - 1/2)}.$$

Thus,  $S_i/c_4(n_i)$  is unbiased for  $\sigma$ . Then we can easily show that  $\bar{S}/c_4(n_k)$  is unbiased for  $\sigma$ , where

$$\bar{S} = \frac{1}{m} \sum_{i=1}^m S_i.$$

Thus, by substituting  $\hat{\mu} = \bar{\bar{X}}$  and  $\hat{\sigma} = \bar{S}/c_4(n)$  into (1), we have the control limits

$$\begin{aligned}\text{UCL} &= \bar{\bar{X}} + \frac{g}{\sqrt{n}} \frac{\bar{S}}{c_4(n)} = \bar{\bar{X}} + A_3(n)\bar{S}, \\ \text{CL} &= \bar{\bar{X}}, \\ \text{LCL} &= \bar{\bar{X}} - \frac{g}{\sqrt{n}} \frac{\bar{S}}{c_4(n)} = \bar{\bar{X}} - A_3(n)\bar{S},\end{aligned}$$

where  $A_3(n) = A(n)/c_4(n) = g/\{c_4(n)\sqrt{n}\}$ .

It is also known that

$$E(R) = d_2(n)\sigma,$$

where  $R$  is the sample range from  $X_i \sim N(\mu, \sigma^2)$  and

$$d_2(n) = 2 \int_0^\infty \left\{ 1 - [\Phi(z)]^n - [1 - \Phi(z)]^n \right\} dz.$$

For more details on  $d_2(n)$ , one can refer to the vignette below.

```
> vignette("factors.cc", package="rQCC")
```

Then, with the  $i$ th sample,  $R_i/d_2(n)$  is unbiased for  $\sigma$ , where

$$R_i = \max_{1 \leq j \leq n} (X_{ij}) - \min_{1 \leq j \leq n} (X_{ij}).$$

Then, with the  $m$  samples,  $\bar{R}/d_2(n)$  is unbiased for  $\sigma$ , where

$$\bar{R} = \frac{1}{m} \sum_{i=1}^m R_i.$$

Substituting  $\hat{\mu} = \bar{\bar{X}}$  and  $\hat{\sigma} = \bar{R}/d_2(n)$  into (1), we have the control limits

$$\begin{aligned}\text{UCL} &= \bar{\bar{X}} + \frac{g}{\sqrt{n}} \frac{\bar{R}}{d_2(n)} = \bar{\bar{X}} + A_2(n)\bar{R}, \\ \text{CL} &= \bar{\bar{X}}, \\ \text{LCL} &= \bar{\bar{X}} - \frac{g}{\sqrt{n}} \frac{\bar{R}}{d_2(n)} = \bar{\bar{X}} - A_2(n)\bar{R},\end{aligned}$$

where  $A_2(n) = A(n)/d_2(n) = g/\{d_2(n)\sqrt{n}\}$ .

As alternatives to the above, we can use robust estimates of location and scale. For example, using the median, we can estimate  $\mu$

$$\hat{\mu} = \frac{M_1 + M_2 + \cdots + M_m}{m} = \frac{1}{m} \sum_{i=1}^m M_i,$$

where

$$M_i = \operatorname{median}_{1 \leq j \leq n}(X_{ij}).$$

One can also consider estimating  $\sigma$  based on the conventional MAD (median absolute deviation) given by

$$\text{MAD} = \frac{\operatorname{median}_{1 \leq i \leq n}|X_i - M|}{\Phi^{-1}(3/4)} \approx 1.4826 \cdot \operatorname{median}_{1 \leq i \leq n}|X_i - M|,$$

where  $X_i \sim N(\mu, \sigma^2)$  and  $M = \operatorname{median}(X_i)$ . Here  $\Phi^{-1}(3/4)$  is needed to make this estimator Fisher-consistent [6] for the standard deviation under the normal distribution. For more details, see the references [7, 8]. It should be noted that the above conventional MAD estimator is Fisher-consistent but not unbiased. The “*unbiased MAD*” (uMAD) with a finite sample is developed by Park, Kim and Wang [8] and implemented in the rQCC package (see `mad.unbiased` function).

Then, with the  $m$  samples, we have the robust unbiased estimate of  $\sigma$  as follows

$$\hat{\sigma} = \frac{\text{uMAD}_1 + \text{uMAD}_2 + \cdots + \text{uMAD}_m}{m} = \frac{1}{m} \sum_{i=1}^m \text{uMAD}_i,$$

where

$$\text{uMAD}_i = \operatorname{uMAD}_{1 \leq j \leq n}(X_{ij}).$$

The `rcc` function constructs the control charts based on various *unbiased* estimates. For example, with the median and uMAD estimates, one can obtain the control limits using the following

```
> rcc(data, loc="median", scale="mad")
```

Another way of constructing the control limits is to use the Hodges-Lehmann [9] for location and Shamos [10] for scale which are respectively given by

$$\text{HL} = \operatorname{median}\left(\frac{X_i + X_j}{2}\right)$$

and

$$\text{Shamos} = \frac{\operatorname{median}_{i < j}(|X_i - X_j|)}{\sqrt{2} \Phi^{-1}(3/4)} \approx 1.048358 \cdot \operatorname{median}_{i < j}(|X_i - X_j|),$$

where  $\sqrt{2} \Phi^{-1}(3/4)$  is needed to make Shamos estimator Fisher-consistent for the standard deviation under the normal distribution [11]. For the Hodges-Lehmann estimate, the median is obtained by three ways: (i) the pairwise averages with  $i < j$  (denoted by HL1), (ii) the pairwise averages with  $i \leq j$  (HL2), and (iii) all the pairwise averages (HL3). For more details, refer to [8]. It should be noted that the above Shamos is Fisher-consistent but not unbiased. The Hodges-Lehmann and “*unbiased Shamos*” are

also developed by [8] and implemented in R (see `HL` and `shamos.unbiased`). For example, with the `HL2` and unbiased Shamos estimates, one can obtain the control limits as below.

```
> rcc(data, loc="HL2", scale="shamos")
```

As shown above, by choosing the options for `loc` and `scale`, one can construct various control charts.

### 3 The $S$ chart

In order to construct the  $CL \pm g \cdot SE$  control limits, we can consider the relation

$$\frac{S_k - E(S_k)}{SE(S_k)} = \pm g.$$

Since  $E(S_k) = c_4(n)\sigma$  and  $SE(S_k) = \sqrt{1 - c_4(n)^2} \cdot \sigma$ , we have

$$E(S_k) \pm g \cdot SE(S_k) = \{c_4(n) \pm g\sqrt{1 - c_4(n)^2}\}\sigma.$$

The control limits for the  $S$  chart are given by

$$\begin{aligned} \text{UCL} &= B_6(n)\sigma, \\ \text{CL} &= c_4(n)\sigma, \\ \text{LCL} &= B_5(n)\sigma, \end{aligned}$$

where

$$\begin{aligned} B_5(n) &= \max\{c_4(n) - g \cdot \sqrt{1 - c_4(n)^2}, 0\}, \\ B_6(n) &= c_4(n) + g \cdot \sqrt{1 - c_4(n)^2}. \end{aligned}$$

Since  $\sigma$  is unknown in practice, we need to choose an appropriate unbiased estimate for  $\sigma$ . One can consider  $\hat{\sigma} = \bar{S}/c_4(n)$ . Then we have

$$\begin{aligned} \text{UCL} &= B_4(n)\bar{S}, \\ \text{CL} &= \bar{S}, \\ \text{LCL} &= B_3(n)\bar{S}, \end{aligned}$$

where  $B_3(n) = B_5(n)/c_4(n)$  and  $B_4(n) = B_6(n)/c_4(n)$ .

To obtain the robustness property, one can consider a robust estimate of  $\sigma$ . For example, the unbiased MAD or unbiased Shamos estimates of  $\sigma$  can be used as seen before. The limits for the  $S$  chart are calculated using the `rcc` function with `type="S"` as below.

```
> rcc(data, scale="mad", type="S")
> rcc(data, scale="shamos", type="S")
```

## 4 The $R$ chart

We consider the relation

$$\frac{R_k - E(R_k)}{\text{SE}(R_k)} = \pm g.$$

Since  $E(R_k) = d_2(n)\sigma$  and  $\text{Var}(R_k) = d_3(n)^2\sigma^2$ , we have

$$E(R_k) \pm g \cdot \text{SE}(R_k) = \{d_2(n) \pm g d_3(n)\}\sigma.$$

The control limits for the  $R$  chart are given by

$$\begin{aligned} \text{UCL} &= D_2(n)\sigma, \\ \text{CL} &= d_2(n)\sigma, \\ \text{LCL} &= D_1(n)\sigma, \end{aligned}$$

where

$$\begin{aligned} D_1(n) &= \max\{d_2(n) - g \cdot d_3(n), 0\}, \\ D_2(n) &= d_2(n) + g \cdot d_3(n), \end{aligned}$$

Since  $\sigma$  is unknown in practice, we need to choose an appropriate unbiased estimate for  $\sigma$ . One can consider  $\hat{\sigma} = \bar{R}/d_2(n)$ . Then we have

$$\begin{aligned} \text{UCL} &= D_4(n)\bar{R}, \\ \text{CL} &= \bar{R}, \\ \text{LCL} &= D_3(n)\bar{R}, \end{aligned}$$

where  $D_3(n) = D_1(n)/d_2(n)$  and  $D_4(n) = D_2(n)/d_2(n)$ . These limits are easily calculated using the `rcc` function as below.

```
> rcc(data, scale="range", type="R")
```

As afore-mentioned, we can consider a robust estimate of  $\sigma$ . For example, the control limits with the unbiased Shamos are calculated as below.

```
> rcc(data, scale="shamos", type="R")
```

## References

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